PMM U.S.S.R., Vol.54,No.4,pp.449-457,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

ASYMPTOTIC LAMINAR WAKES*

N.I. YAVORSKII

The multipole approach, developed in the theory of laminar non-selfsimilar submerged streams /1, 2/, is applied to the problem of the uniform motion of a body in a space flooded with an incompressible viscous liquid. The zeroth approximation is an exact solution of the complete equations of hydrodynamics which is valid in a neighbourhood of the point at infinity and represents the exact asymptotic structure of the solution of the boundary-value problem. One such solution for a non-selfsimilar submerged stream is the Landau solution /3/. In the context of the laminar wake behind a body, this solution corresponds to prescribing a constant velocity v_0 and pressure p_0 at infinity (the body is assumed to be at rest). According to the multipole approach /1, 2/, the basis for the generalized multipole expansion is the solution of the Navier-Stokes equations, linearized in terms of this exact asymptotic solution.

Expressing the solution of the complete Navier-Stokes equations in the neighbourhood of the point at infinity as

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{w}, \; p = p_0 + q, \quad \mathbf{w} = o(|\mathbf{v}_0|) \tag{0.1}$$

we obtain equations for the perturbations $\mathbf{w}, q$ (with the density of the liquid assumed equal to unity):

$$\begin{gathered} \nu\Delta\mathbf{w} - (\mathbf{v}_0, \nabla)\mathbf{w} - \nabla q = (\mathbf{w}, \nabla)\mathbf{w}, \quad \operatorname{div}\mathbf{w} = 0 \\ \mathbf{w} = \mathbf{w}_*, \quad \mathbf{x} \in \Sigma \; (\mathbf{w}_* = \mathbf{v}_*(\mathbf{s}) - \mathbf{v}_0, \quad \mathbf{s} \in \Sigma) \end{gathered} \tag{0.2}$$

where $\mathbf{w}_*$ is the prescribed velocity on the surface Σ of the body.

To a first approximation, since $\mathbf{w} = O(v_0)$, we can neglect the non-linear term on the right of (0.2), thus obtaining the well-known Oseen equations /4/:

$$\nu\Delta\mathbf{w}_0 - (\mathbf{v}_0, \nabla)\mathbf{w}_0 - \nabla q = 0, \quad \operatorname{div}\mathbf{w}_0 = 0 \tag{0.3}$$

If the general solution of Eqs.(0.3) is known, one can construct a solution satisfying the prescribed boundary conditions on the surface of the body. Then, by successive approximations to the non-linear term in (0.2), one obtains a corresponding asymptotic expansion at infinity.

Oseen's fundamental solution of Eqs.(0.3) took the form of a "velocity tensor" $E_{ij}(\mathbf{x}-\mathbf{y})$ and pressure vector $e_i(\mathbf{x}-\mathbf{y})$ /5/:

$$E_{ij} = \delta_{ij}\Delta\Phi - \frac{\partial^2\Phi}{\partial x_i \partial x_j}, \quad e_i = -\frac{\partial}{\partial x_i}[\nu\Delta\Phi - (\mathbf{v}_0, \nabla\Phi)] \tag{0.4}$$

$$\Phi = -\frac{1}{8\pi\sigma\nu}\int_0^{\sigma s}\frac{1-e^{-\alpha}}{\alpha}\,d\alpha, \quad 2\nu\sigma = |\mathbf{v}_0|, \quad s = |\mathbf{x}-\mathbf{y}| - \mathbf{n}_0\cdot(\mathbf{x}-\mathbf{y}), \; \mathbf{n}_0 = \frac{\mathbf{v}_0}{|\mathbf{v}_0|}$$

The solution of Oseen's problem (0.3) with the velocity $\mathbf{w}_0 = \mathbf{w}_*(\mathbf{s})$ prescribed on the surface of the body may be determined from a system of Fredholm integral equations /6, 7/, derivable with the aid of the fundamental solution (0.4).

In the context of the full Navier-Stokes equations, existence /8, 9/ and uniqueness /8/ theorems in the flow problem have been established for Reynolds numbers $\mathrm{Re} = v_0r_0/\nu$ (r_0 being the characteristic dimension of the body) in a certain neighbourhood of zero: $\mathrm{Re} \in [0, \mathrm{Re}_*]$, $\mathrm{Re}_* > 0$. An existence theorem has been proved for generalized solutions, with no restriction on the Reynolds number, for steady flow around n bodies, on the assumption that the total outflux from the submerged bodies is zero. The behaviour of the solution as $|\mathbf{x}| = r \to \infty$ is of interest. It has been shown /8/ that if $|\mathbf{w}(\mathbf{x})| \leqslant Cr^{-\alpha}, \alpha > 1/2, C > 0$, then at some distance from the body

**Prikl.Matem.Mekhan.*, 54,4,543-552,1990

$$v_i(\mathbf{x}) = v_{0i} + a_j E_{ij}(\mathbf{x}) + b e_i(\mathbf{x}) + \sigma_i(\mathbf{x})$$
$$\sigma_i(\mathbf{x}) = O(r^{-3/2}\ln(\sigma r)), \quad a_i, b = \text{const} \tag{0.5}$$

Formally speaking, the expansion (0.5) involves no restriction on the Reynolds number, the only essential condition being that $r \gg r_0$. It has been shown /10/ that the representation (0.5) holds for any flow situation in which the Dirichlet integral of the fields $\mathbf{v}(\mathbf{x}), p(\mathbf{x})$ is finite. Now, in view of the fact that thanks to the existence theorems the Dirichlet integral is bounded, one can assume that (0.5) holds for any steady flow. Further estimates have been obtained /11/:

$$\sigma_i(\mathbf{x}) = v_i^{3/2}(\mathbf{x}) + O(r^{-2}\ln^3(\sigma r)), \quad v_i^{3/2} = O(r^{-3/2}\ln(\sigma r)) \tag{0.6}$$

It will be shown in this paper that the first terms of the asymptotic expansion of the velocity and pressure field may be expressed in terms of the exact conservation integrals: momentum flux, mass flux, flux of angular momentum and higher moments of these quantities. Explicit expressions will be obtained for the second and third terms of the expansion, which are determined by these quantities. Further terms of the expansion are not so universal and depend on the specific formulation of the boundary-value problem for the flow. A previous asymptotic expansion of the vorticity /11/ is refined and an estimate is obtained for the remainder term.

1. We start from certain integral relations which can be obtained from the Navier-Stokes Eqs.(0.1), (0.2) by inverting the Oseen operator (0.3); these relations may be written as follows:

$$w_i(\mathbf{x}) = \int_E w_i w_j \frac{\partial}{\partial x_k} E_{ij}(\mathbf{x}-\mathbf{y})\, d^3y +$$
$$\int_\Sigma \{E_{ij}(\mathbf{x}-\mathbf{y})(v_{0j}v_k - \Pi_{jk}) + w_j T_{ijk}(\mathbf{x}-\mathbf{y})\} n_k dS \tag{1.1}$$

$$q(\mathbf{x}) = \int_E w_i w_j \frac{\partial}{\partial x_k} e_j(\mathbf{x}-\mathbf{y})\, d^3y +$$
$$\int_\Sigma \{e_j(\mathbf{x}-\mathbf{y})(v_{0j}v_k - \Pi_{jk}) + w_j T_{jk}(\mathbf{x}-\mathbf{y})\} n_k dS \tag{1.2}$$

$$T_{ijk} = \nu\left(\frac{\partial E_{ij}}{\partial x_k} + \frac{\partial E_{ik}}{\partial x_j}\right) - e_i\delta_{jk}, \quad T_{jk} = \nu\left(\frac{\partial e_j}{\partial x_k} + \frac{\partial e_k}{\partial x_j}\right) - e^*\delta_{jk}$$
$$\Pi_{ij} = v_i v_j + p\delta_{ij} - \nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j}\right) \tag{1.3}$$

where Π_{ij} is the momentum flux tensor, E is the volume occupied by the liquid, Σ is the surface of the body, $\mathbf{n}$ is the outward normal, $E_{ij}(\mathbf{x}), e_i(\mathbf{x}), e^*(\mathbf{x}) = (\mathbf{v}_0, \nabla(1/r))$ is a fundamental solution of Oseen's problem (0.4).

The integral representation (1.1)-(1.3) forms the basis for the multipole approach proposed here.

The volume integrals in (1.1), (1.2) exhibit the following asymptotic structure as $r \to \infty$ /8/:

$$I_i(\mathbf{x}) = \int_E w_i w_j \frac{\partial}{\partial x_k} E_{ij}(\mathbf{x}-\mathbf{y})\, d^3y = O(r^{-3/2}\ln(\sigma r)) \tag{1.4}$$
$$I(\mathbf{x}) = \int_E w_i w_k \frac{\partial}{\partial x_k} e_j(\mathbf{x}-\mathbf{y})\, d^3y = O(r^{-3}\ln(\sigma r))$$

while that of the surface integrals is

$$K_i(\mathbf{x}) = \int_\Sigma \{E_{ij}(\mathbf{x}-\mathbf{y})(v_{0j}v_k - \Pi_{jk}) + w_j T_{ijk}(\mathbf{x}-\mathbf{y})\} n_k dS = O(1/r) \tag{1.5}$$
$$K(\mathbf{x}) = \int_\Sigma \{e_j(\mathbf{x}-\mathbf{y})(v_{0j}v_k - \Pi_{jk}) + w_j T_{jk}(\mathbf{x}-\mathbf{y})\} n_k dS = O(1/r^2)$$

Since the body is bounded and may be enclosed in a sphere Ω_0 of radius r_0, it follows that for sufficiently large $r = |\mathbf{x}|$, when $|\mathbf{y}| \leqslant r_0$, the functions E_{ij}, e_j in the integrands of (1.5) may be expanded in absolutely convergent Taylor series. Substituting these series into (1.5), we obtain (summation over n from 1 to ∞)

$$K_i(\mathbf{x}) = a_j E_{ij}(\mathbf{x}) + \Sigma a_{jk_1\ldots k_n} \frac{\partial^n E_{ij}(\mathbf{x})}{\partial x_{k_1}\ldots\partial x_{k_n}} + b e_i(\mathbf{x}) + \Sigma b_{k_1\ldots k_n} \frac{\partial^n e_i(\mathbf{x})}{\partial x_{k_1}\ldots\partial x_{k_n}} \tag{1.6}$$

$$K(\mathbf{x}) = a_j e_j(\mathbf{x}) + \sum a_{jk_1\ldots k_n} \frac{\partial^n e_j(\mathbf{x})}{\partial x_{k_1}\ldots\partial x_{k_n}} + be^*(\mathbf{x}) + \sum b_{k_1\ldots k_n} \frac{\partial^n e^*(\mathbf{x})}{\partial x_{k_1}\ldots\partial x_{k_n}} \tag{1.7}$$

$$a_{jk_1\ldots k_n} = \frac{(-1)^n}{n!} \oint_\Sigma [(v_{0j}v_l - \Pi_{jl})\, n_l y_{k_1} - nv\,(w_j n_{k_1} + w_{k_1} n_j)]\, y_{k_2}\ldots y_{k_n} dS$$

$$b_{k_1\ldots k_n} = \frac{(-1)^{n+1}}{n!} \oint_\Sigma w_l n_l y_{k_1}\ldots y_{k_n} dS \tag{1.8}$$

The quantities a_i, b may be expressed as

$$\mathbf{a} = Q\mathbf{v}_0 - \mathbf{J}, \quad b = -Q \tag{1.9}$$

where J is the total momentum flux from the body and Q is the ouflow of liquid from the body.

It can be shown tht the expansions (1.6), (1.7) are convergent, since w_i and Π_{ij} are bounded on the surface of the body. Note that the series (1.6), (1.7) are constructed in such a way that each successive term in the sums is small to a higher order in $1/r$ than its predecessor. This may be proved on the basis of the asymptotic estimates $\nabla^n E_{ij} = O(r^{-1-n/2})$, $\nabla e_i = O(r^{-n-2})$.

In view of the analogy with the Laplace equation, we call (1.6) a multipole expansion of the solution of the steady flow problem for the body. In that case the quantities $a_{jk_1\ldots k_n}$, $b_{k_1\ldots k_n}$ are the strengths of the appropriate n-th-order multipoles. It should be mentioned that the expansion we have constructed is not a simple corollary of Oseen's equations, as might be expected for a solution of the Navier-Stokes equations at $r \gg r_0$. The coefficients of the expansions (1.8) are non-linear functions of the velocity. In order to determine $a_{jk_1\ldots k_n}$, therefore, we need a solution of the entire non-linear boundary-value problem of the flow.

Taking into account that the volume integrals (1.4) become asymptotically small compared with the surface integrals (1.5) as $r\to\infty$, we can construct a full expansion of the solution of the problem by successive approximations to the non-linearity involved in the volume integrals $I_i(\mathbf{x})$, $I(\mathbf{x})$. A natural choice for the zeroth approximation is

$$\mathbf{w}^\circ(\mathbf{x}) = \mathbf{K}(\mathbf{x}), \quad q^\circ(\mathbf{x}) = K(\mathbf{x}) \tag{1.10}$$

i.e., the additive part of the full solution (1.1), (1.2) corresponding to the surface integrals. This is the approach underlying the existence and uniqueness proofs in /8/, where analogous approximations are shown to converge to a solution of the boundary-value problem if the Reynolds number Re is sufficiently small. Thus, this path leads to the construction of an exact solution of the Navier-Stokes equation for the flow problem as a multipole expansion at infinity.

2. Let us assume that the solution $\mathbf{w}^\circ(\mathbf{x}), q^\circ(\mathbf{x})$ is known. This is equivalent to specification of all the multipole strengths $a_{jk_1\ldots k_n}$, $b_{k_1\ldots k_n}$ (a denumerable number of constants). Eqs.(1.3) and (1.4) may be written, taking (1.7) and (1.9) into account as follows:

$$w_i(\mathbf{x}) = w_i^\circ(\mathbf{x}) + \int_E w_i w_j \frac{\partial}{\partial x_k} E_{ij}(\mathbf{x}-\mathbf{y})\, d^3y \tag{2.1}$$

$$q(\mathbf{x}) = q^\circ(\mathbf{x}) + \int_E w_i w_k \frac{\partial}{\partial x_k} e_j(\mathbf{x}-\mathbf{y})\, d^3y$$

The first equation of (2.1) is autonomous, and it will therefore suffice to apply successive approximations to this equation only, obtaining a system of recurrence relations

$$w_i(\mathbf{x}) = \sum_{n=0}^{\infty} W_i^n(\mathbf{x}) \tag{2.2}$$

$$w_i^{n+1}(\mathbf{x}) = \sum_{n=0}^{\infty} \int_E w_j^l w_k^{n-l} \frac{\partial}{\partial x_k} E_{ij}(\mathbf{x}-\mathbf{y})\, d^3y, \quad n = 0, 1, 2, \ldots$$

It should be noted that the construction of a solution in this form does not yet yield a sequence of terms of increasing order with respect to $1/r$; hence different approach is necessary to construct the expansion at infinity in closed form.

Since the integral operators in (2.1) are convolutions, it is clearly convenient to change to Fourier transforms. Let $W_i(\mathbf{k})$, $P(\mathbf{k})$, $A_{ij}(\mathbf{k})$ denote the Fourier transforms of $w_i(\mathbf{x})$, $q(\mathbf{x})$, $E_{ij}(\mathbf{x})$, respectively. Putting $w_i(\mathbf{x}) \equiv 0$ for $\mathbf{x} \in B^0$, where B is the volume occupied by the body and B^0 its interior, we have

$$W_i(\mathbf{k}) = \int\limits_{E^3\setminus B^\circ} e^{i\mathbf{k}\cdot\mathbf{x}} w_i(\mathbf{x})\, d^3x, \quad E^3\setminus B^\circ = E$$

$$A_{ij}(\mathbf{k}) = \int\limits_{E^3} e^{i\mathbf{k}\cdot\mathbf{x}} E_{ij}(x)\, d^3x = \frac{k_i k_j - k^2\delta_{ij}}{\nu k^2 [k^2 - 2i\sigma(\mathbf{k}\cdot\mathbf{n}_0)]}$$

The Fourier transforms of Eqs.(2.1) are

$$\begin{gathered} W_l(\mathbf{k}) = W_l^0(\mathbf{k}) - iA_{lm}(\mathbf{k})\, k_n B_{mn}(\mathbf{k}) \\ P(\mathbf{k}) = P^0(\mathbf{k}) - k_n k_m k^{-2} B_{mn}(\mathbf{k}) \\ B_{ij}(\mathbf{k}) = \int\limits_{E^3\setminus B^\circ} e^{i\mathbf{k}\cdot\mathbf{x}} w_i(\mathbf{x})\, w_j(\mathbf{x})\, d^3x = (2\pi)^{-3} \int\limits_{K^3} W_i(\mathbf{p})\, W_j(\mathbf{k}-\mathbf{p})\, d^3p \end{gathered} \tag{2.3}$$

This system of equations is considerably simpler than the original system (2.1), though still involving integral equations. The functions $W_i^\circ(\mathbf{k})$, $P^c(\mathbf{k})$ are given by

$$\begin{gathered} W_l^\circ(\mathbf{k}) = A_{lj}(\mathbf{k})\, M_k(\mathbf{k}) - ik_l k^{-2} M(\mathbf{k}) \\ P^\circ(\mathbf{k}) = -ik_l k^{-2} M_l(\mathbf{k}) - ik_l v_{0l} \mathbf{k}^{-2} M(\mathbf{k}) \end{gathered} \tag{2.4}$$

where $M_j(\mathbf{k})$, $M(\mathbf{k})$ are analytic functions in a neighbourhood of $k = 0$:

$$\begin{gathered} M_j(\mathbf{k}) = a_j + \sum_{n=1}^{\infty} (-i)^n a_{jl_1\ldots l_n} k_{l_1} \ldots k_{l_n} \\ M(\mathbf{k}) = b + \sum_{n=1}^{\infty} (-i)^n b_{l_1\ldots l_n} k_{l_1} \ldots k_{l_n} \end{gathered} \tag{2.5}$$

Comparing the structure of expansions (1.3), (1.7) with (2.4) and (2.5), we note that the terms in each are of corresponding orders of magnitude as $r \to \infty$ and as $\mathbf{k} \to 0$. Thus, expanding the tensor $B_{ij}(k)$ in series as $\mathbf{k} \to 0$ in (2.3) will produce the necessary full multipole expansion of the solution in the Fourier transform space.

According to a theorem of Finn /12/, the kinetic energy of the perturbed motion, i.e., the kinetic energy determined using $\mathbf{w}(\mathbf{x})$, is infinite. This means that $B_{ii}(0) = \infty$. Consequently the tensor $B_{ij}(\mathbf{k})$ is a non-analytic function at $\mathbf{k} = 0$. Nevertheless, one can construct the required expansion of $B_{ij}(\mathbf{k})$ as $\mathbf{k} \to \infty$, by isolating the singularity. The type of singularity can be determined by using the successive approximations (2.2), having first changed to their Fourier transforms:

$$\begin{gathered} W_j^{n+1}(\mathbf{k}) = -iA_{jl}(\mathbf{k})\, k_m B_{lm}^n(\mathbf{k}) \\ B_{ij}^n(\mathbf{k}) = (2\pi)^{-3} \sum_{l=0}^{n} \int\limits_{K^3} W_i^{n-l}(\mathbf{p})\, W_j^l(\mathbf{k}-\mathbf{p})\, d^3p \end{gathered}$$

3. Responsibility for the divergence of the kinetic energy integral lies with the principal terms of the expansion as $r \to \infty$ (0.5), since, as the velocity field $w_i(\mathbf{x})$ is bounded, divergence may be observed only at the upper end of the interval, when $r = \infty$. Taking this into consideration, let us consider the integral

$$B_{ij}^{\circ\circ}(k) = a_k a_l \int\limits_{K^3} A_{ij}(\mathbf{p})\, A_{jl}(\mathbf{k}-\mathbf{p})\, d^3p \tag{3.1}$$

where the zeroth approximation is the Fourier transform of the principal term of the expansion $a_j E_{ij}(\mathbf{x})$.

Define a tensor

$$R_{ijkl}(\mathbf{k}) = \int\limits_{K^3} A_{ij}(\mathbf{p})\, A_{kl}(\mathbf{k}-\mathbf{p})\, d^3p = \int\limits_{K^3} \frac{(\delta_{ij} - m_i m_j)(\delta_{kl} - \tau_k\tau_l)}{(p^2 - 2i\sigma p_3)[(\mathbf{k}-\mathbf{p})^2 - 2i\sigma(k_3 - p_3)]}\, d^3p \tag{3.2}$$

$$m_i = \frac{p_i}{p}, \quad \tau_i = \frac{k_i - p_i}{|\mathbf{k}-\mathbf{p}|}, \quad n_i = \frac{k_i}{k}, \quad \mathbf{v}_0 = (0, 0, v_0)$$

The p_3 axis points in the direction of the velocity $\mathbf{v}_0$. Since

$$|\delta_{ij} - m_i m_j|, \quad |\delta_{kl} - m_k m_l| \leqslant 1$$

it follows that, in order to determine the type of singularity of $R_{ijkl}(\mathbf{k})$ as $\mathbf{k} \to 0$, it will suffice to consider the integral

$$\alpha(\mathbf{k}) = \int_{K^3} \{(p^2 - 2i\sigma p_3)[(\mathbf{k}-\mathbf{p})^2 - 2i\sigma p\,(k_3 - p_3)]\}\, d^3p \tag{3.3}$$

Let

$$\begin{gathered} \mathbf{m} = (\sin\theta\cos\varphi, \quad \sin\theta\sin\varphi, \ \cos\theta) \\ \mathbf{n} = (\sin\theta_0\cos\varphi_0, \quad \sin\theta_0\sin\varphi_0, \ \cos\theta_0) \\ p_3 = p\cos\theta, \quad k_3 = k\cos\theta_0 \end{gathered} \tag{3.4}$$

Then (3.3) may be written as

$$\alpha(\mathbf{k}) = \int_0^\infty p\,dp \int_{-1}^{1} \frac{dx}{p - 2i\sigma x} \int_0^{2\pi} \frac{d\varphi}{c_1 + c_2\cos(\varphi - \varphi_0)} \tag{3.5}$$

$$\begin{gathered} c_1 = p^2 + k^2 - 2i\sigma k x_0 + 2p\,(i\sigma - k x_0)\,x \\ c_2 = -2kp\sqrt{(1 - x_0^2)(1 - x^2)}, \quad x = \cos\theta, \quad x_0 = \cos\theta_0 \end{gathered} \tag{3.6}$$

If either $k \ll \sigma,\ x_0 \neq 0$ or $1 - x_0^2 \ll 1$, this will be valid at a distance in the wake region, since the latter is paraboloid in shape. After integration of (3.5) by parts, we get

$$a(\mathbf{k}) = -\frac{\pi^2}{4\sigma}\ln\frac{k^2 - 2i\sigma k x_0}{8\sigma^2} + \frac{\pi c_0}{2\sigma}\sqrt{\frac{k^2 - 2i\sigma k x_0}{8\sigma^2}} + O(k) \tag{3.7}$$

$$c_0 = \int_0^\infty \frac{\ln(1 + y^2)}{y^2}\, dy$$

Thus, the tensor $B_{ij}(\mathbf{k})$ has a logarithmic singularity as $\mathbf{k} \to \infty$. This improves on Finn's theorem, according to which the integral of the kinetic energy of the perturbed motion is divergent; Finn's theorem here results as a corollary of (3.7). Note that the expansion (3.6) involves not only a logarithmic term but also fractional (semi-integral) powers of k, a circumstance which should have significant influence on the expansion of the solution in a series of type (1.6), (1.7). To obtain such expansions it will be necessary to use the apparatus of fractional differentiation /13/.

In order to get a better idea of the behaviour of the tensor $R_{ijkl}(\mathbf{k})$ as $\mathbf{k} \to \infty$, let us write it as

$$R_{ijkl}(\mathbf{k}) = \alpha(\mathbf{k})\,\delta_{ij}\delta_{kl} - \alpha_{ij}(\mathbf{k})\,\delta_{kl} - \alpha_{kl}(\mathbf{k})\,\delta_{ij} - \alpha_{ijkl}(\mathbf{k}) \tag{3.8}$$

$$\begin{gathered} \alpha(\mathbf{k}) = \int_{K^3} \frac{d^3p}{F(\mathbf{k}, \mathbf{p})}, \quad \alpha_{ij}(\mathbf{k}) = \int_{K^3} \frac{m_i m_j}{F(\mathbf{k}, \mathbf{p})}\, d^3p \\ \alpha_{ijkl}(\mathbf{k}) = \frac{m_i m_j \tau_k \tau_l}{F(\mathbf{k}, \mathbf{p})}\, d^3p, \quad \alpha_{ijkl} = \alpha_{klij} \\ F(\mathbf{k}, \mathbf{p}) = (p^2 - 2i\sigma p_3)\,[(\mathbf{k} - \mathbf{p})^2 - 2i\sigma\,(k_3 - p_3)] \end{gathered} \tag{3.9}$$

The quantity $\alpha(\mathbf{k})$ is just (3.3), so it has already been investigated. The representation (3.8),(3.9) follows from (3.2) by using the properties of convolutions.

Consider the tensor

$$\alpha_{ij}(\mathbf{k}) = \int_0^\infty p\,dp \int_{-1}^{1} \frac{dx}{p - 2i\sigma} \int_0^{2\pi} \frac{m_i m_j\, d\varphi}{c_1 + c_2\cos(\varphi - \varphi_0)}, \quad (x = \cos\theta) \tag{3.10}$$

If $k \ll \sigma,\ x_0 \neq 0$, it follows from (3.6) that the number c_2 in the denominator of the integrand in (3.10) may be ignored. Integration with respect to φ and x yields the expression

$$\alpha_{ij}(\mathbf{k}) = \alpha(\mathbf{k})\,\delta_{ij}\lambda_{(j)} + \frac{\pi^2}{4\sigma}\,\delta_{ij}\nu_{(j)} + O(k^{1/2}) \tag{3.11}$$
$$\lambda = (1/2,\ 1/2,\ 0), \quad \nu = (-1,\ -1,\ 2)$$

Consider the tensor $\alpha_{ijkl}(\mathbf{k})$. Using the definition of α_{ijkl} in (3.9), it can be shown that

$$\alpha_{ijkl}(\mathbf{k}) = \mathring{\alpha}_{ijkl}(\mathbf{k}) + O(k), \quad \mathring{\alpha}_{ijkl}(\mathbf{k}) = \int_{K^3} \frac{m_i m_j m_k m_l}{F(\mathbf{k},\mathbf{p})}\, d^3p \tag{3.12}$$

As when calculating $\alpha_{ij}(\mathbf{k})$, consideration of (3.6) shows that if $k \ll \sigma$, $x_0 \neq 0$ then the expression

$$\mathring{\alpha}_{ijkl}(\mathbf{k}) = \int_0^\infty p\,dp \int_{-1}^{1} \frac{dx}{p - 2i\sigma x} \int_0^{2\pi} \frac{m_i m_j m_k m_l}{c_1 + c_2\cos(\varphi - \varphi_0)}\, d\varphi \tag{3.13}$$

may be simplified by neglecting c_2

Put

$$m_{ijkl} = \int_0^{2\pi} m_i m_j m_k m_l\, d\varphi \tag{3.14}$$

It can be shown that the only non-zero components m_{ijkl} will be $m_{(ij)(ij)}$, $m_{(ij)(ij)}$, $m_{(ii)(jj)}$ (no summation over repeated indices!). Direct calculation gives

$$m_{ijkl} = M_{(ij)}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + M_{(jk)}\delta_{ij}\delta_{kl} \tag{3.15}$$
$$M_{11} = M_{22} = 3(1 - x^2)^2/8, \quad M_{33} = x^4, \quad x = \cos\theta$$
$$M_{11} = M_{21} = (1 - x^2)^2/8, \quad M_{13} = M_{31} = M_{23} = M_{32} = 1/2x^2(1 - x^2)$$

Integrating, we deduce from (3.13)-(3.15) that

$$\mathring{\alpha}_{ijkl}(\mathbf{k}) = \left[\alpha(\mathbf{k})\Lambda_{(ij)} + \frac{\pi^2}{4\sigma}N_{(ij)}\right](\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + \tag{3.16}$$
$$\left[\alpha(\mathbf{k})\Lambda_{(jk)} + \frac{\pi^2}{4\sigma}N_{(jk)}\right]\delta_{ij}\delta_{kl} + O(k^{1/2})$$
$$\Lambda_{11} = \Lambda_{22} = {}^3/_8, \ \Lambda_{12} = \Lambda_{21} = {}^1/_8, \ \Lambda_{13} = \Lambda_{31} = \Lambda_{23} = \Lambda_{32} = \Lambda_{33} = 0$$
$$N_{11} = N_{22} = -{}^9/_8, \ N_{12} = N_{21} = -{}^3/_8, \ N_{13} = N_{31} = N_{23} = N_{32} =$$
$${}^1/_2, \quad N_{33} = 1$$

Finally, from (3.1), (3.2), (3.8), (3.11), (3.12) and (3.16), we have

$$\mathring{\mathring{B}}_{ij}(\mathbf{k}) = \frac{\alpha(\mathbf{k})}{(2\pi)^3\nu^2}G_{ij}(\mathbf{a}) + \frac{1}{32\pi\sigma\nu^2}K_{ij}(\mathbf{a}) + O(k^{1/2}) \tag{3.17}$$
$$G_{11} = \frac{9a_1^2 + a_2^2}{8}, \quad G_{12} = G_{21} = \frac{a_1a_2}{4}, \quad G_{13} = G_{31} = \frac{a_1a_3}{2}$$
$$G_{22} = \frac{a_1^2 + 9a_2^2}{8}, \quad G_{33} = a_3^2$$
$$K_{11} = \frac{-11a_1^2 - 3a_2^2 + 4a_3^2}{8}, \quad K_{12} = K_{21} = -\frac{5a_1a_2}{4}$$
$$K_{22} = \frac{-3a_1^2 - 11a_2^2 + 4a_3^2}{8}$$
$$K_{13} = K_{31} = K_{23} = K_{32} = 0, \quad K_{33} = \frac{a_1^2 + a_2^2 - 2a_3^2}{2}$$

The second and last approximations to the non-linearity do not produce logarithmically divergent terms, but they all make a contribution of $O(1)$ to the term. Hence we have the following expansion:

$$B_{ij}(\mathbf{k}) = -\frac{1}{32\pi\sigma\nu^2}G_{ij}(\mathbf{a})\,ln\,\frac{k^2 - 2i\sigma k x_0}{8\sigma^2} + \frac{1}{32\pi\sigma\nu^2}K_{ij}(\mathbf{a}) + \mathring{p}_{ij} + O(k^{1/2}) \tag{3.18}$$

where p_{ij}° is the renormalized mean momentum flux tensor of the perturbed motion:

$$\text{Real } \mathring{p}_{ij} = \mathring{P}_{ij} = \int_E (w_i w_j - a_k a_l E_{ik} E_{jl})\, d^3x \tag{3.19}$$

4. As already pointed out, the asymptotic behaviour of the solution of the problem of the flow around a body may be obtained by asymptotic expansion of the Fourier transform $W_i(\mathbf{k})$ of the velocity of the liquid at $\mathbf{k} \to \infty$. Substituting the asymptotic representation of the tensor $B_{ij}(\mathbf{k})$ (3.18) into Eq.(2.6) and using (2.9), (2.11), we get

$$\begin{gathered} W_l(\mathbf{k}) = A_{lm}(\mathbf{k}) \{a_m - ik_n G_{mn}(\mathbf{a})(2\pi)^{-3}\nu^{-2} a(\mathbf{k}) - ik_n [a_{mn} + \\ {}^1/_4\pi^2\sigma^{-1}\nu^{-2} K_{mn}(\mathbf{a}) + p_{mn}^{\circ}]\} - ik_l k^{-2}(b - ik_n b_n) + O(k^{3/2}) \end{gathered} \tag{4.1}$$

Since the factor $-ik_j$ introduces a differentiation $\partial/\partial x_j$ into the inverse transform, it follows that to obtain the asymptotic expansion of $w_i(\mathbf{x})$ we need only investigate the Fourier transforms of the functions $A_{ij}(\mathbf{k})\,\alpha(\mathbf{k})$. The function $A_{ij}(\mathbf{k})$ is the Fourier transform of the velocity tensor $E_{ij}(\mathbf{x})$.

Let $U_{ij}(\mathbf{x})$ be the tensor whose Fourier transform is $A_{ij}(\mathbf{k})\,\alpha(\mathbf{k})$. Using the formula for $A_{ij}(\mathbf{k})$ we can write the tensor $U_{ij}(\mathbf{x})$ as

$$U_{ij} = \delta_{ij}\Delta U - \partial^2 U/\partial x_i \partial x_j \tag{4.2}$$

where $U(\mathbf{x})$ is the inverse Fourier transform of

$$D(\mathbf{k}) = a(\mathbf{k})\,[\nu k^2(k^2 - 2i\sigma\mathbf{k}\cdot\mathbf{n}_0)]^{-1}$$

Hence follows the conclusion that

$$L\Delta U = \chi(\mathbf{x}), \quad L = \Delta - 2\sigma(\mathbf{n}_0, \nabla) \tag{4.3}$$

where $\chi(\mathbf{x})$ is the inverse Fourier transform of

$$h(\mathbf{k}) = \alpha(\mathbf{k})/\nu$$

Relation (3.3) for $a(\mathbf{k})$ yields

$$\alpha(\mathbf{k}) = \int d^3p H(\mathbf{k}) H(\mathbf{k} - \mathbf{p}), \quad H(\mathbf{k}) = \frac{1}{k^2 - 2i\sigma\mathbf{k}\cdot\mathbf{n}_0}$$

Hence, changing to inverse Fourier transforms, we get

$$\Delta F - 2\sigma(\mathbf{n}_0, \nabla F) = -\delta(\mathbf{x}), \quad F(\mathbf{x}) = \frac{e^{-\sigma s}}{4\pi r}$$

$$F(\mathbf{x}) = \frac{1}{(2\pi)^3} \int H(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^3k$$

and

$$\chi(\mathbf{x}) = \frac{1}{(2\pi^2)^3} \int n(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^3k = (2\pi^2)^3 F^2(\mathbf{x}) = \frac{\pi e^{-2\sigma s}}{2\nu r^2} \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\Delta U = f, \quad \Delta f - 2\sigma(\mathbf{n}_0, \nabla f) = {}^1/_2\pi e^{-2\sigma s}/(\nu r^2) \tag{4.5}$$

Investigating the solution of Eq.(4.5) by separation of variables, one can show that

$$\begin{gathered} f(\mathbf{x}) = -\frac{\pi}{4\nu\sigma} \frac{e^{-\sigma s}\ln(\sigma r)}{r} - \frac{2\pi^2}{r} e^{-\sigma s}\Phi(s) - \\ \frac{\pi}{4\nu\sigma} \frac{e^{-\sigma s}}{r} e^{2\sigma r} \int_{\sigma r}^{\infty} \frac{e^{-2\alpha}}{\alpha}\, d\alpha + f_0(\mathbf{x}) \end{gathered} \tag{4.6}$$

where $f_0(\mathbf{x})$ is the solution of the homogeneous Eq.(4.5). The term $f_0(\mathbf{x})$ is not taken into consideration any more, since it is actually contained in the solution for the velocity field in the terms corresponding to the expansion of the surface integral.

In analogous fashion, integrating the equation $\Delta U = f$, we obtain the asymptotic expansion

$$U(\mathbf{x}) = \frac{\pi^2}{\sigma}\Phi(s)\ln(\sigma r) - \pi^2 \int_0^s \frac{d\alpha}{\alpha}\int_0^\alpha e^{-\sigma\beta}\Phi(\beta)\,d\beta + \frac{\pi^2}{2\sigma r}\left[2\int_0^s \Phi\,ds - \int_0^s \frac{d\alpha}{\alpha}\int_0^\alpha e^{-\sigma\beta}\Phi(\beta)\,d\beta\right] + \sum_{n=2}^{\infty} S_n(s)\,r^{-n} \tag{4.7}$$

Further terms of the series follow from the equation

$$(sS_n')' = (n-1)\,[2sS_{n-1} + (n-2)\,S_{n-1}''],\ n \geqslant 2 \tag{4.8}$$

which is recursively solvable. The solution of the homogeneous Laplace equation is omitted from (4.7), for the reason indicated above. Using formulae (4.2), (4.7), one can obtain an asymptotic expansion of the tensor $U_{ij}(\mathbf{x})$.

We now turn to the Fourier transform $W_i(\mathbf{k})$ (4.1). Again taking inverse transforms, we find an asymptotic expansion of the velocity field:

$$w_i(\mathbf{x}) = a_j E_{ij}(\mathbf{x}) + \left(b + b_j\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_i}\frac{1}{4\pi r} + \frac{1}{32\pi\sigma\nu^2}G_{jl}(\mathbf{a})\frac{\partial}{\partial x_l}U_{ij}(\mathbf{x}) + \left[a_{jl} + \frac{1}{32\pi\sigma\nu^2}K_{jl}(\mathbf{a}) + \overset{\circ}{P}_{jl}\right]\frac{\partial}{\partial x_l}E_{ij}(\mathbf{x}) + O\left(\frac{\ln^3\sigma r}{r^2}\right) \tag{4.9}$$

The remainder term in (4.9) was in fact determined in /11/, since it may in fact be obtained without the additional assumption adopted here that the vectors **a** and $\mathbf{v}_0$ are collinear.

It is of interest to develop an asymptotic formula for the vorticity field. We first observe that the "gradient terms" $\partial U_{ij}^2/\partial x_i \partial x_j$ in U_{ij} and $\partial\Phi_{ij}^2/\partial x_i\partial x_j$ in E_{ij} (0.4), (4.2), make no contribution to rot **v**. The term $\delta_{ij}\Delta U$ is determined by formula (4.6), and $\Delta\Phi = -e^{-\sigma s}/(4\pi\nu r)$. Hence one obtains the asymptotic expansion of the vorticity:

$$\omega_i(\mathbf{x}) = \frac{\sigma}{4\pi\nu}e_{ijk}\frac{\partial s}{\partial x_j}a_k\frac{e^{-\sigma s}}{r} + e_{ijk}\frac{G_{jl}(\mathbf{a})}{32\pi\sigma\nu^2}\frac{\partial^2 f}{\partial x_k\partial x_l} - e_{ijk}\left[a_{jl} + \frac{1}{32\pi\sigma\nu^2}K_{jl}(\mathbf{a}) + \overset{\circ}{P}_{jl}\right]\frac{\partial^2}{\partial x_l\partial x_k}\frac{e^{-\sigma s}}{4\pi\nu r} + O\left(\frac{\ln^3\sigma r}{r^{5/2}}e^{-\sigma(1-\varepsilon)s}\right),\quad 0 < \varepsilon \ll 1 \tag{4.10}$$

where e_{ijk} is the antisymmetric Levi-Civita tensor.

It is characteristic that all the terms of this expression decay exponentially $\sim e^{-\sigma s}$ outside the wake ($s \to \infty$). An estimate for the remainder term may be obtained by using a result from /11/, according to which the exponential decay of vorticity outside the wake obeys the law $\omega \sim \varphi(r)\,e^{-\sigma(1-\varepsilon)s}$, as well as the estimate of (4.9) for the velocity remainder term; in the derivation it must be remembered that differentiation of the fundamental tensors E_{ij}, U_{ij}, etc., with respect to the coordinate contributes no more than the factor $(s/r)^{1/2}$ $((\nabla s)^2 = 2s/r)$ to the estimate. For a rigorous justification of the remainder term in (4.10), one can use Propositions 2 and 3 from /11/. From (4.10) one obtains the following refinement of the estimate for $\omega_i(\mathbf{x})$ derived in /11/:

$$\boldsymbol{\omega}(\mathbf{x}) = \frac{\sigma}{4\pi\nu}(\nabla s \times \mathbf{a})\frac{e^{-\sigma s}}{r} + O((1 + \sigma s)\,e^{-\sigma s}r^{-2}\ln(\sigma r)) \tag{4.11}$$

It should be noted that the asymptotic expansion presented in /14/ does not predict exponential decay of vorticity outside the wake. This is because the asymptotic expression obtained for the second term of the expansion in /14/ is very coarse. Our present approach avoids this pitfall and yields all the terms of the asymptotic expression in terms of exact conservation integrals.

We have thus obtained an asymptotic expansion for the velocity field for the laminar wake at some distance from a body of arbitrary shape immersed in a liquid, improving on the results of /11, 14/. Explicit expressions have been obtained for the second and third terms of the asymptotic expansion, developed - like the first term (0.5) - in terms of conservation integrals. This makes the representation (4.9) universal and enables one to determine the three principal terms of the asymptotic expansion for the wake at a distance, given the characteristics of the flow in the vicinty of the body, and conversely, to determine various specific characteristics pertaining to the body, given the velocity distribution in the wake. In particular, using the first three terms of the expansion one can obtain not only the force of drag but also the drag coefficient, which is of no little importance in certain practical applications. On the other hand, the fact that the principal terms of the expansion are expressed in terms of exact conservation integrals enables one to apply it not only to problems of flow past a body, which are characterized by a momentum sink, but also to various other hydrodynamic problems involving sources of momentum, such as the distribution of the stream in a wake, a stream inclined at an angle to the flow, the motion of a body with reactive thrust, etc.

REFERENCES

1. GOL'DSHTIK M.A. and YAVORSKII N.I., On submerged streams. Prikl. Mat. Mekhan., 50, 4, 1986.
2. YAVORSKII N.I., Non-axisymmetric submerged streams. Prikl. Mat. Mekh., 52, 5, 1988.
3. LANDAU L.D., On a new exact solution of the Navier-Stokes equations. Dokl. Akad. Nauk SSSR, 43, 7, 1944.
4. OSEEN C.W., Über die Stokes Formel, und über eine verwandte Aufgabe in der Hydrodynamik. Ark. Mat. Astronom. Fys., 6, 11, 1910.
5. OSEEN C.W., Neuere Methoden und Ergebnisse in der Hydrodynamik. Leipzig. Akademische Verlagsgesellschaft, 1927.
6. FAXEN H., Fredholmsche Integralgleichungen zur der Hydrodynamik zäher Flüssigkeiten. Ark. Mat. Astronom. Fys., A21, 14, 1928/29.
7. OSEEN C.W., Das hydrodynamische Randwertproblem. Z. angew. Math. Mech., 10, 4, 1930.
8. FINN R., On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems. Arch. Rat. Mech. Anal., 19, 5, 1965.
9. LADYZHENSKAYA O.A., Mathematical Problems in Viscous Fluid Dynamics, Nauka, Moscow, 1970.
10. BABENKO K.I., On stationary solutions of the problem of the flow of a viscous incompressible liquid around a body, Mat. Sbornik, 91, 1, 1973.
11. BABENKO K.I. and VASIL'YEV M.M., On the asymptotic behaviour of the stationary flow of a viscous liquid at a distance from a body. Prikl. Mat. Mekh., 37, 4, 1973.
12. FINN R., Stationary solutions of the Navier-Stokes Equations, Inst. Gidrodin, Sibirsk. Otd. Akad. Nauk SSSR, Novosibirsk, 1967.
13. SAMKO S.G., KILBAS A.A. and MARICHEV O.I., Integrals and Derivatives of Fractional Order and Some of their Applications, Nauka i Tekhnika, Minsk, 1987.
14. VASIL'YEV M.N., On the asymptotic behaviour of the velocity and forces acting on a body in stationary flow of a viscous liquid. Prikl. Mat. Mekh., 38, 1, 1974.

Translated by D.L.